

Midterm Examination with Answers

Trick or Treat: Due 2:30 pm, Tuesday, October 31

Instructions

This exam has one main question, with two parts. You have four hours total to answer it. It should not take that long if you think first.

The second question is to state whether you think we should keep papers on the reading list. This part of the exam is not timed.

All three parts carry equal weight.

You may not collaborate on this examination. You may use any notes that you have personally taken, downloaded from the official course web site, or copied by hand (in pencil or ink on paper). You may use any book in the Caltech library collection, if you think it will help. You may use Mathematica or a similar program, but you must explain what you asked it to do and why.

Write legibly in complete sentences and explain yourself.

1 Variations on a theme

Recall the simple growth model with two factors of production, capital K and labor L , constant returns to scale, and no technical change or depreciation, just population growth. It says

$$Y(t) = F(K(t), L(t)),$$

and

$$L(t) = L_0 e^{nt},$$

where n is the percentage rate of growth of the labor supply.

Let $k(t) = K(t)/L(t)$ denote the capital/labor ratio and $y(t) = Y(t)/L(t)$ denote the output per capita. Because of constant returns to scale we may write

$$y = f(k), \quad \text{where } f(k) = F(k, 1).$$

We assume that a constant fraction s of output is invested in creating new capital, so

$$\dot{K}(t) = sY(t) = sf(k(t))L(t). \quad (1)$$

We can also write $K(t) = k(t)L(t)$, which gives another expression for $\dot{K}$,

$$\dot{K}(t) = \dot{k}(t)L(t) + k(t)nL(t). \quad (2)$$

Equating (1) and (2) implies that the capital/labor ratio evolves according to the differential equation

$$\dot{k}(t) = sf(k(t)) - nk(t). \quad (3)$$

For a Cobb–Douglas production function,

$$Y = K^\alpha L^{1-\alpha}, \quad (4)$$

per capita income is given by

$$f(k) = k^\alpha. \quad (5)$$

The solution to the differential equation (3) for the Cobb–Douglas case is

$$k(t) = \left[\left(k_0^{1-\alpha} - \frac{s}{n} \right) e^{-n(1-\alpha)t} + \frac{s}{n} \right]^{\frac{1}{1-\alpha}}, \quad (6)$$

so that

$$k(t) \rightarrow \left(\frac{s}{n} \right)^{\frac{1}{1-\alpha}}. \quad (7)$$

You may wish to consult the on-line notes at <http://people.hss.caltech.edu/~kcb/Courses/EC140/Notes/SolowGrowth.pdf>, which has an appendix on differential equations.

1.1 Variation I

Assume that there are three factors of production, physical capital K , human capital H , and raw labor L , and there is a Cobb–Douglas production function

$$Y = K^\alpha H^\eta L^{1-\alpha-\eta},$$

where $\alpha > 0$, $\eta > 0$, and $1 - \alpha - \eta > 0$. The rate of saving is fixed at s , but now it may be invested in either human or physical capital. Assume that at each instant t , the marginal product of K and H are the same. (You can justify this by assuming that investors will invest in whichever form of capital has the higher marginal product, and that the history is such that the marginal products are already equalized.)

1. What is the marginal product of K and of H ?
2. What does this assumption of equality about marginal products imply about the relation between K and H ?
3. Finish the analysis of the dynamics of this model. By this we mean that you should answer the following questions
 - (a) Is there a nonzero steady state? That is, is there a value k^* such that $k(t) \equiv k^*$ satisfies the differential equation (3)?
 - (b) If so, what is it? On what parameters does it depend? Is it unique? Is it stable?
 - (c) If not, does growth in $y(t)$ continue positively, or is it negative?
 - (d) Ideally, you should write down a closed a form solution for $k(t)$, if you can.

Part of the answer: A differential equation

It follows from (3), (5), and (6) above that:

The solution to the differential equation

$$\dot{x}(t) = ax(t)^b - cx(t) \quad (8)$$

where $a, b, c > 0$, $b < 1$, with initial condition $x(0) = x_0$ is

$$x(t) = \left[\left(x_0^{1-b} - \frac{a}{c} \right) e^{-c(1-b)t} + \frac{a}{c} \right]^{\frac{1}{1-b}} \quad (9)$$

Consequently, there is a nonzero steady state

$$x^* = \left(\frac{a}{c} \right)^{\frac{1}{1-b}} \quad (10)$$

and

$$x(t) \xrightarrow[t \rightarrow \infty]{} \left(\frac{a}{c} \right)^{\frac{1}{1-b}} \quad (11)$$

provided $x_0 > 0$.

Sample answer

1. For the Cobb–Douglas case, the marginal product of physical capital, $\partial Y / \partial K$ is just

$$\alpha Y / K,$$

and the marginal product of human capital, $\partial Y / \partial H$ is just

$$\eta Y / H.$$

2. Equating the marginal products implies

$$\alpha / K = \eta / H \quad \text{or} \quad H = \frac{\eta}{\alpha} K.$$

3. This means that we may write

$$Y = F\left(K, \frac{\eta}{\alpha}K, L\right) = \left(\frac{\eta}{\alpha}\right)^\eta K^{\alpha+\eta} L^{1-\alpha-\eta},$$

which is the same as (4) except that the exponent on K is now $\alpha + \eta$ and there is a factor $(\eta/\alpha)^\eta$ out front. The analysis is now straightforward. In particular, (3) becomes

$$\dot{k}(t) = \sigma k(t)^{\alpha+\eta} - nk(t),$$

where

$$\sigma = s\left(\frac{\eta}{\alpha}\right)^\eta.$$

As a result, this model's behavior can be written down by replacing a by σ , b by $\alpha + \eta$, and c by n in equation (8). Consequently the model possesses a unique stable nonzero steady state level of k at

$$k^* = \left(\frac{\sigma}{n}\right)^{\frac{1}{1-\alpha-\eta}},$$

so the steady state of y and c are

$$y^* = \left(\frac{\eta}{\alpha}\right)^\eta \left(\frac{\sigma}{n}\right)^{\frac{\alpha+\eta}{1-\alpha-\eta}}, \quad c^* = (1-s)y^*.$$

(Can we expand σ to simplify this in a useful way?)

1.2 Variation II

Assume that there are three factors of production, physical capital K , land A , and raw labor L , and there is a Cobb-Douglas production function

$$Y = K^\alpha A^\gamma L^{1-\alpha-\gamma},$$

where $\alpha > 0$, $\gamma > 0$, and $1 - \alpha - \gamma > 0$. The quantity of land is fixed and unalterable.

1. What is the crucial difference between this variation and the one above?
2. Analyze the dynamics of this model.

Sample answer

1. The main difference between this variation and the previous variation is that land is fixed and investment cannot increase the quantity of land, but we can still try to see if we can still somehow follow the program in the introduction to the problem to get a differential equation of the form (8).
2. We start by rewriting this puppy as

$$\begin{aligned} Y(t) &= K(t)^\alpha A^\gamma L(t)^\lambda, \\ \dot{K}(t) &= sY(t), \\ L(t) &= L_0 e^{nt}, \end{aligned}$$

where $\lambda = 1 - \alpha - \gamma$. You might be tempted to think that we can simply write

$$Y = \hat{F}(K, L) = A^\gamma K^\alpha L^\lambda,$$

which is absolutely correct. The problem is that $\hat{F}$ does not exhibit constant returns to scale, so while the above is true, we cannot use it the same way. Instead, recall the discussion in class with labor-augmenting technological change, and let's define a variable $Z(t)$ so that

$$Y = \hat{F}(K, Z) = K^\alpha Z^{1-\alpha} = A^\gamma K^\alpha L^\lambda,$$

which does exhibit constant returns to scale in K and Z . It is clear that Z must be

$$Z(t) = A^{\frac{\gamma}{\gamma+\lambda}} L(t)^{\frac{\lambda}{\gamma+\lambda}}, \quad (12)$$

which we may call the **land-handicapped labor**. Then

$$Y(t) = K(t)^\alpha Z(t)^{\gamma+\lambda}, \quad (13)$$

which does exhibit constant returns to scale in K and Z . Define

$$\hat{y}(t) = \frac{Y(t)}{Z(t)}, \quad \kappa(t) = \frac{K(t)}{Z(t)},$$

the land-handicapped income per capita and capital/land-handicapped labor ratio. Then (13) implies

$$\hat{y}(t) = \kappa(t)^\alpha. \quad (14)$$

Now we may write

$$\dot{K}(t) = sY(t) = s\kappa(t)^\alpha Z(t). \quad (15)$$

We also know

$$K(t) = \kappa(t)Z(t),$$

so

$$\dot{K}(t) = \dot{\kappa}(t)Z(t) + \kappa(t)\dot{Z}(t). \quad (16)$$

To compute $\dot{Z}$, from (12) we know

$$\begin{aligned} \dot{Z}(t) &= A^{\frac{\gamma}{\gamma+\lambda}} \frac{\lambda}{\gamma+\lambda} \frac{L(t)^{\frac{\lambda}{\gamma+\lambda}}}{L(t)} \dot{L}(t) \\ &= A^{\frac{\gamma}{\gamma+\lambda}} \frac{\lambda}{\gamma+\lambda} L(t)^{\frac{\lambda}{\gamma+\lambda}} \frac{\dot{L}(t)}{L(t)} \\ &= A^{\frac{\gamma}{\gamma+\lambda}} \frac{\lambda}{\gamma+\lambda} n L(t)^{\frac{\lambda}{\gamma+\lambda}}, \end{aligned}$$

which implies

$$\begin{aligned} \frac{\dot{Z}(t)}{Z(t)} &= \frac{A^{\frac{\gamma}{\gamma+\lambda}} \frac{\lambda}{\gamma+\lambda} n L(t)^{\frac{\lambda}{\gamma+\lambda}}}{A^{\frac{\gamma}{\gamma+\lambda}} L(t)^{\frac{\lambda}{\gamma+\lambda}}} \\ &= \frac{n\lambda}{\gamma+\lambda}. \end{aligned}$$

So back to (16), using $\dot{Z} = \frac{\dot{Z}}{Z}Z$, and combining (15) and (16), we get

$$\dot{\kappa}(t) = s\kappa(t)^\alpha - \frac{n\lambda}{\gamma+\lambda}\kappa(t),$$

which sure looks a lot like (8), with

$$x = \kappa, \quad a = s, \quad b = \alpha, \quad \text{and } c = \frac{n\lambda}{\gamma+\lambda}.$$

Thus we know that κ converges to a steady-state value

$$\kappa^* = \left(\frac{(\gamma + \lambda)s}{\lambda n} \right)^{\frac{1}{\gamma + \lambda}},$$

(recall that $1 - \alpha = \gamma + \lambda$), so $\hat{y}$ converges to the steady-state value

$$\hat{y}^* = \left(\frac{(\gamma + \lambda)s}{\lambda n} \right)^{\frac{\alpha}{\gamma + \lambda}},$$

but $\hat{y}$ is not per capita income. Instead,

$$\hat{y}(t) = \frac{Y(t)}{L(t)} \left(\frac{L}{A} \right)^{\frac{\gamma}{\gamma + \lambda}}.$$

Thus convergence of $\hat{y}(t) \rightarrow \hat{y}^*$ implies

$$\frac{Y(t)}{L(t)} \sim y^* A^{\frac{-\gamma}{\gamma + \lambda}} L_0 e^{-\frac{n\gamma}{\gamma + \lambda} t} \rightarrow 0.$$

This is bad news!