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Growth in a Cobb–Douglas Overlapping Generations Model

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1 Basic formulation and notation

This describes the basic OG model used by Auerbach and Kotlikoff [1].

1. There are N members of each generation, and they are identical—the **representative agent hypothesis**. Each works when young, saves, and lives off his or her savings when old.

2. At the beginning of period t, there is a capital stock K_t, which is owned by the older generation (generation $t-1$), that is used with an amount of labor L_t supplied by the young of period t to produce and additional increment Y_t of good by the end of the period. This means that at the end of period t the total amount available for consumption and investment is $Y_t + K_t$.

3. Workers of generation t work 1 period at real wage w_t. They save an amount a_{t+1}, which earns the real rate of return r_{t+1}.

4. National output Y is produced according to the production function

$$Y_t = F(K_t, L_t)$$

which exhibits constant returns to scale. Because of constant returns to scale, the economy acts as if it is maximizing the aggregate real profit

$$F(K_r, L_t) - wL_t - \rho_t K_t,$$

where ρ_t is the rental rate of capital, that is, what must be paid to the owners of capital.

Also because of constant returns to scale, the marginal product of capital and of labor, and the output per worker, depend only the **capital/labor ratio**

$$k_t = \frac{K_t}{L_t}.$$

Indeed

$$\frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}\right) = F(k_t, 1) = f(k_t),$$

where the last equality is taken as the definition of f.

Also from $f(K/L) = F(K/L, 1)$, fixing L and differentiating with respect to K implies

$$\frac{f'(K/L)}{L} = \frac{D_1 F(K/L, 1)}{L}, \qquad \text{so} \qquad f'(k) = D_1 F(k, 1) = D_1 F(K, L),$$

where the last equality follows from Euler's Theorem (partial derivatives are homogeneous of degree zero).

5. Workers are fully employed,

$$L_t = N.$$

6. Wages and interest rates in each period adjust to equate the supply and demand of capital and labor.

2 Consumer/workers

Consumer/workers have a **utility function** for ranking lifetime consumption plans,

$$u(c_y, c_o),$$

where c_y is consumption when young, and c_o is consumption when old. Workers earn wage income w_y when young and w_o when old. Thus their budget constraint is

$$c_y = w_y - a, \qquad \text{and} \qquad c_o = w_o + (1 + r)a,$$

where a is their **saving** and r is the **real rate of interest**. Solving the second constraint for a and substituting into the first gives the equivalent single lifetime budget constraint:

$$c_y + \frac{c_o}{1+r} = w_y + \frac{w_o}{1+r}$$

or

$$\text{present value of consumption} = \text{present value of income}.$$

Consumer/workers maximize their utility subject to the budget constraint. That is, they choose c_y and c_o to

$$\underset{c_y, c_o}{\text{maximize}}\ u(c_y, c_o) \text{ subject to } c_y + \frac{c_o}{1+r} = w_y + \frac{w_o}{1+r}.$$

An equivalent, and perhaps more useful reformulation is that they choose their saving a to

$$\underset{a}{\text{maximize}}\ u(w_y - a, w_o + (1 + r)a).$$

The solution to this problem clearly depends on w_y, w_o, and the real interest rate r. The solution is known as the **saving function**, and we shall denote it by

$$\mathfrak{a}(r; w_y, w_o),$$

so with time subscripts

$$a_{t+1} = \mathfrak{a}(r_{t+1}; w_t, w_{t+1}).$$

2.1 Cobb–Douglas case

We now solve the consumer/worker's problem for the case

$$u(c_y, c_o) = c_y^{\alpha} c_o^{1-\alpha},$$

and

$$w_o = 0.$$

The Lagrangean for this problem is

$$c_y^{\alpha} c_o^{1-\alpha} + \lambda\Big(w_y - c_y - \frac{c_o}{1+r}\Big).$$

The first order conditions for a maximum at (c_y, c_o, a) are that its partial derivatives are zero, or

$$\begin{aligned} \alpha {c_y}^{\alpha-1} {c_o}^{1-\alpha} - \lambda &= 0 \\ (1-\alpha){c_y}^{\alpha} {c_o}^{-\alpha} - \frac{1}{1+r}\lambda &= 0. \end{aligned}$$

Combined with the budget constraint, this implies

$$\begin{aligned} c_y &= \alpha w_y \\ a &= (1-\alpha) w_y \\ c_o &= (1+r)(1-\alpha) w_y. \end{aligned}$$

In terms of the time subscript convention, the key result is the saving function satisfies

$$\boxed{a_{t+1} = \mathfrak{a}(r_{t+1}; w_t) = (1-\alpha) w_t.} \tag{1}$$

Note that for this utility function, the individual saving function is independent of the real interest rate! This will simplify the analysis of the model.

3 The production sector

As remarked above, the production sector acts as if it maximizes

$$F(K, L) - wL - \rho K.$$

The first order conditions are that the partial derivatives of the profit with respect to K and L are zero. Due to constant returns to scale, the maximum profit is zero, and the profit maximizing output is indeterminate. However, the capital/labor ratio is determinate.

3.1 The Cobb–Douglas case

The solution to the problem when

$$F(K,L) = AK^{\beta}L^{1-\beta}$$

is gotten by solving the first order conditions,

$$\begin{aligned}\beta AK^{\beta-1}L^{1-\beta} - \rho &= 0\\ (1-\beta)AK^{\beta}L^{-\beta} - w &= 0,\end{aligned}$$

or in terms of the optimal capital/labor ratio k,

$$\begin{aligned}\beta Ak^{\beta-1} &= \rho\\ (1-\beta)Ak^{\beta} &= w.\end{aligned}$$

These first order conditions allows to solve for k as

$$k = \frac{\beta}{1-\beta}\frac{w}{\rho}.$$

So let us write the **capital demand per worker function** as

$$\boxed{k_t = \mathsf{k}(\rho_t, w_t) = \frac{\beta}{1-\beta}\frac{w_t}{\rho_t}.} \tag{2}$$

4 Some accounting

Now we get to the relation between ρ_t, the rental rate of capital at time t, and r_t, the real rate of interest between periods $t-1$ and t; and the relation between a_t, the per capita saving by the young of generation $t-1$, and k_t, the capital/labor ratio in period t. Here it is:

$$\boxed{\begin{aligned}a_t &= k_t\\ r_t &= \rho_t.\end{aligned}} \tag{3}$$

That is, the capital stock per worker at time t, is exactly the savings per worker of time $t-1$, and the rental rate of capital is the real rate of interest.

Recall that by Euler's Theorem, for any values of K and L, we have

$$F(K,L) = D_1F(K,L)K + D_2F(K,L)L,$$

and for the profit maximizing values K^* and L^* (which depend on ρ and w) we have

$$D_1F(K^*,L^*) = \rho \qquad \text{and} D_2F(K^*,L^*) = w.$$

Thus

$$Y^* = F(K^*, L^*) = \rho K^* + wL^*.$$

Now wL^* is the aggregate income of the young, and ρK^* is the capital rental income of the old. But the old also own the capital stock K^*, which they must either sell to the young at the end of the period, after they have collected their rental fee, or consume it themselves. How much can they get for it? Well, it is K^* units of output, which is worth exactly K^* units of output.

Thus on the income side, the young receive wL_t^* and the old receive ρK_t^* in rental income and K_t^* from the sale of their capital. Thus total income is

$$wL^* + \rho K^* + K^* = Y^* + K^*.$$

But the real rate of interest r_t between periods $t-1$ and t is computed as follows. The young at time $t-1$ save a real amount Na_t^*, for which they pay Na_t^* units output, which they sell a period later for $Na_t^* = K_t^*$ and also receive $\rho_t K_t^*$ in rental income, so they get back $(1+\rho_t)K_t^*$, which by the definition of real rate of interest is the return on their investment $(1+r_t)K_t^*$. Thus

$$r_t = \rho_t.$$

From now on, I will use r instead of ρ.

5 Equilibrium values of w and r

From (1) the saving of the workers, which is the supply of capital, is given by

$$a_{t+1} = \mathsf{a}(r_{t+1}; w_t) = (1-\alpha)w_t,$$

and from (2), the demand for capital is given by

$$k_{t+1} = \mathsf{k}(r_{t+1}, w_{t+1}) = \frac{\beta}{1-\beta}\frac{w_{t+1}}{r_{t+1}}.$$

In equilibrium of supply and demand for capital, we must have a sequence of equilibrium wage and real interest rates

$$\langle (r_t^*, w_t^*) \rangle_{t=0,1,\ldots}$$

and

$$\mathsf{a}(r_{t+1}^*; w_t^*) = \mathsf{k}(r_{t+1}^*, w_{t+1}^*)$$

or

$$\boxed{(1-\alpha)w_t^* = \frac{\beta}{1-\beta}\frac{w_{t+1}^*}{r_{t+1}^*} \quad \text{for all } t.} \tag{4}$$

This is not very easy to interpret, but if we express the results directly in terms of the sequence $\langle k_t^* \rangle$ we have the difference equation

$$\boxed{k_{t+1}^* = (1-\alpha)(1-\beta)A{k_t^*}^{\beta}.} \tag{5}$$

6 The steady states

This difference equation has two steady states. A **steady state** is a value $\hat{k}$ such that the solution $k_t^* = \hat{k}$ for all t solves (5). One (rather uninteresting steady state) is given by

$$k_t^* = 0 \qquad \text{for all } t.$$

The other steady state is given by

$$k_t^* = \bar{k} = [(1-\alpha)(1-\beta)A]^{\frac{1}{1-\beta}} \qquad \text{for all } t.$$

This steady state has the property that for any initial condition $k_0^* > 0$ the solution converges relatively rapidly to $\bar{k}$. (I am rapidly running out of ornaments for my letters.)

To see this, consult Figure 1.

7 Growth

Suppose now that the coefficient of total factor productivity A is growing over time at the rate ζ, that is

$$A_{t+1} = (1+\zeta)A_t,$$

and that population is growing at the rate η. That is,

$$N_{t+1} = (1+\eta)N_t.$$

The per capita production function now depends on the time period:

$$y_t = (1+\zeta)^t A_0 k_t^\beta.$$

Therefore, since the real wage is the marginal product labor, and the rate of return on capital is the marginal product of capital we have

$$w_t = (1-\beta)(1+\zeta)^t A_0 k_t^\beta$$
$$r_t = \beta(1+\zeta)^t A_0 k_t^{\beta-1}.$$

The budgets of consumers are straightforward:

$$c_{\text{y}t} + a_{t+1} = w_t$$
$$c_{\text{o}t+1} = (1+r_{t+1})a_{t+1},$$

so the lifetime budget constraint is

$$c_{\text{y}t} + \frac{1}{1+r_{t+1}} c_{\text{o}t+1} = w_t.$$

This gives the demand function

$$c_{\text{y}t} = \alpha w_t$$

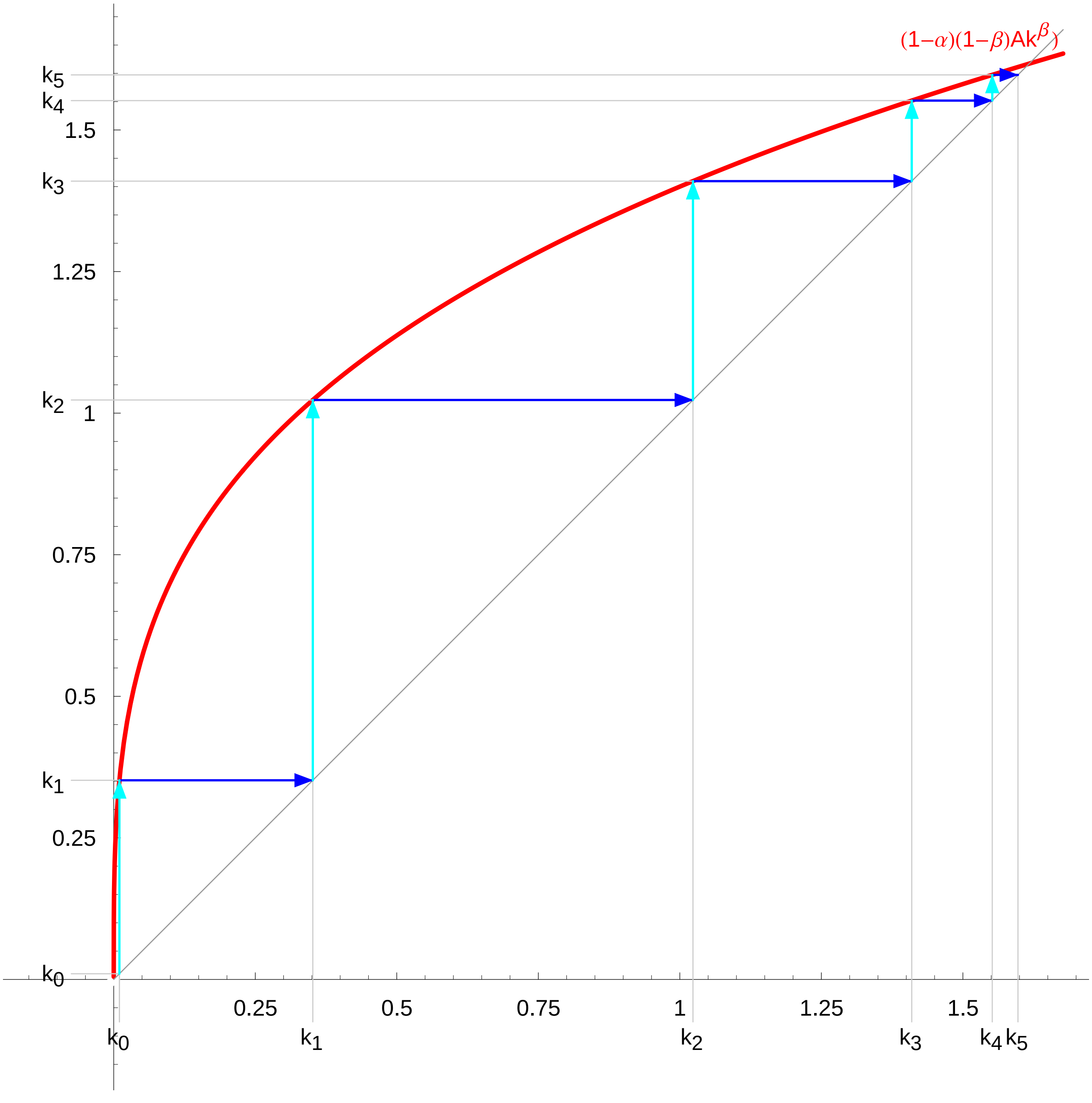

Figure 1. Convergence to the steady-state capital-labor ratio.

so per capita saving a_{t+1} is given by

$$a_{t+1} = (1-\alpha)w_t.$$

A significant difference is that since the population N_{t+1} at time $t+1$ is $(1+\eta)N_t$. Therefore the capital/labor ratio k_{t+1} at time $t+1$ satisfies the **transition equation**

$$\begin{aligned} k_{t+1} &= \frac{a_{t+1}}{1+\eta} \\ &= \frac{(1-\alpha)w_t}{1+\eta} \\ &= \frac{(1-\alpha)(1-\beta)(1+\zeta)^t A_0}{1+\eta} k_t^\beta. \end{aligned} \tag{6}$$

Figure 2 shows some sample dynamics. It is clear that no steady state capital/labor ratio exists, but it may not be immediately obvious that

$$\boxed{\frac{k_{t+1}}{k_t} \xrightarrow[t\to\infty]{} (1+\zeta)^{1/(1-\beta)}.} \tag{$\star$}$$

Before proving ($\star$), let me make a couple of observations. The interesting thing about this limit is what it does not depend on. *The limiting rate of growth in k_t is independent of the initial capital/labor ratio, the rate of growth of population η, and also independent of the saving rate* $1-\alpha$! It depends only on the technological parameters β, the share of capital, and ζ, the rate of productivity growth. Note that the limiting growth rate is increasing in both ζ and β.

The other thing to note is that per capita income and the real wage also grow at the same rate asymptotically:

$$\frac{y_{t+1}}{y_t} = \frac{(1+\zeta)^t A_0 k_t^\beta}{(1+\zeta)^t A_0 k_t^\beta} = (1+\zeta)\left(\frac{k_{t+1}}{k_t}\right)^\beta \to (1+\zeta)(1+\zeta)^{\frac{\beta}{1-\beta}} = (1+\zeta)^{\frac{1}{1-\beta}},$$

$$\frac{w_{t+1}}{w_t} = \frac{(1-\beta)(1+\zeta)^t A_0 k_t^\beta}{(1-\beta)(1+\zeta)^t A_0 k_t^\beta} = (1+\zeta)\left(\frac{k_{t+1}}{k_t}\right)^\beta \to (1+\zeta)(1+\zeta)^{\frac{\beta}{1-\beta}} = (1+\zeta)^{\frac{1}{1-\beta}}.$$

The rate of the return on capital, however, stabilizes:

$$\frac{r_{t+1}}{r_t} = \frac{\beta(1+\zeta)^t A_0 k_t^{\beta-1}}{\beta(1+\zeta)^t A_0 k_t^{\beta-1}} = (1+\zeta)\left(\frac{k_{t+1}}{k_t}\right)^{\beta-1} \to (1+\zeta)(1+\zeta)^{\frac{\beta-1}{1-\beta}} = 1.$$

To prove ($\star$), from (6) we have

$$\begin{aligned} k_{t+1} &= \frac{(1-\alpha)(1-\beta)(1+\zeta)^{t-1} A_0}{1+\eta} k_t^\beta \\ k_t &= \frac{(1-\alpha)(1-\beta)(1+\zeta)^t A_0}{1+\eta} k_{t-1}^\beta, \end{aligned}$$

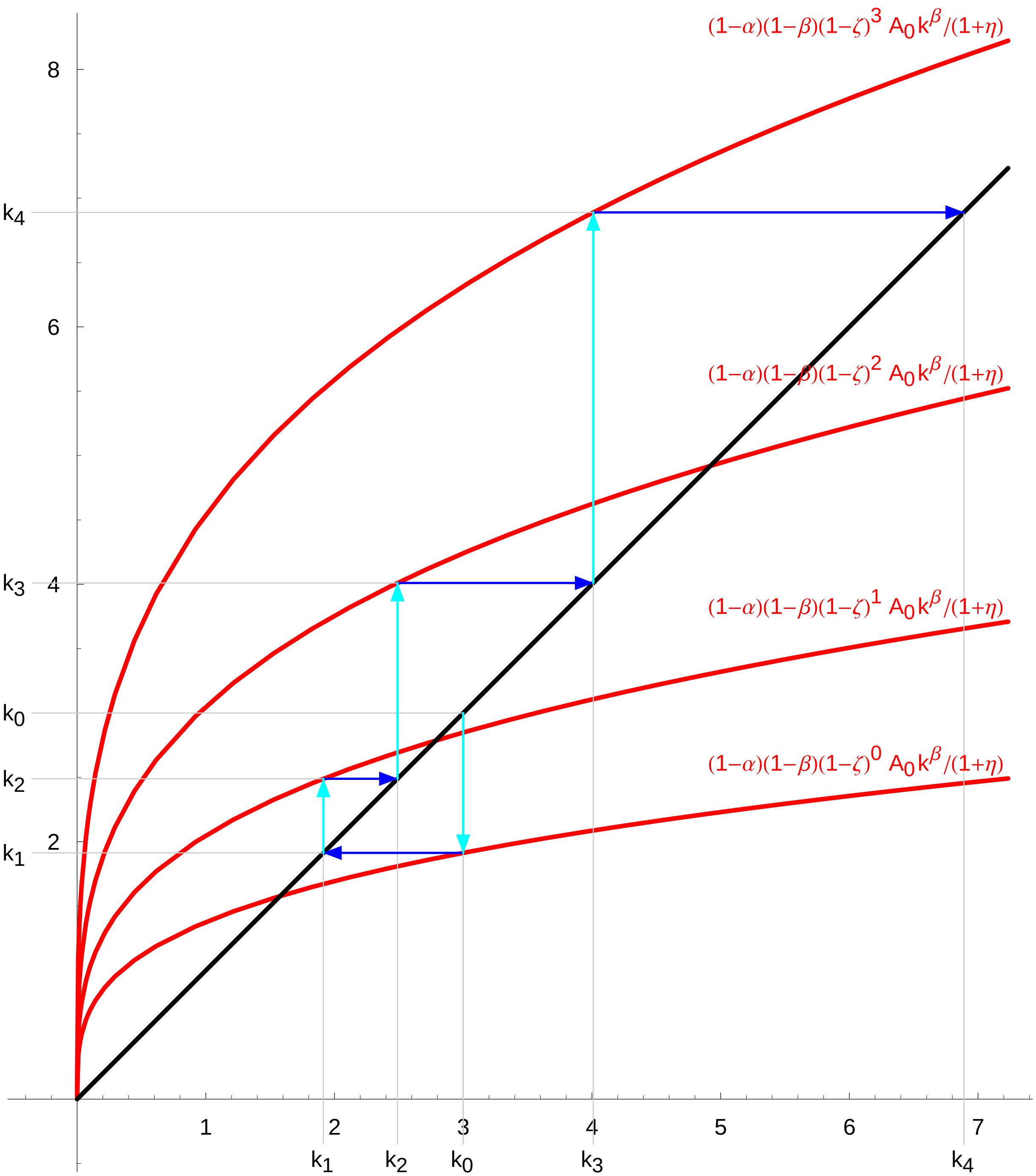

Figure 2. Sample time path of k_t. Note that in this case it is not monotonic.

which implies

$$\frac{k_{t+1}}{k_t} = (1+\zeta)\left(\frac{k_t}{k_{t-1}}\right)^{\beta}.$$

Take logarithms and make the substitution

$$u_t = \ln\left(\frac{k_{t+1}}{k_t}\right)$$

to get the difference equation

$$u_t = \ln(1+\zeta) + \beta u_{t-1}.$$

We can solve this by iterating:

$$\begin{aligned}
u_t &= \ln(1+\zeta) + \beta u_{t-1} \\
&= \ln(1+\zeta) + \beta\big(\ln(1+\zeta) + \beta u_{t-1}\big) \\
&= (1+\beta)\ln(1+\zeta) + \beta^2 u_{t-1} \\
&= (1+\beta)\ln(1+\zeta) + \beta^2\big(\ln(1+\zeta) + \beta u_{t-2}\big) \\
&= (1+\beta+\beta^2)\ln(1+\zeta) + \beta^3 u_{t-2} \\
&\;\;\vdots \\
&= \left(\sum_{n=0}^{t}\beta^n\right)\ln(1+\zeta) + \beta^{t+1}u_0 \\
&\xrightarrow[t\to\infty]{} \frac{\ln(1+\zeta)}{1-\beta}.
\end{aligned} \tag{7}$$

Thus

$$\ln\left(\frac{k_{t+1}}{k_t}\right) \xrightarrow[t\to\infty]{} \frac{\ln(1+\zeta)}{1-\beta} + 0,$$

so exponentiating,

$$\frac{k_{t+1}}{k_t} \xrightarrow[t\to\infty]{} (1+\zeta)^{\frac{1}{1-\beta}}.$$

Note that the special value k_0^* defined by

$$(1+\zeta)^{1/(1-\beta)}k_0^* = k_1 = \frac{(1-\alpha)(1-\beta)(1+\zeta)^0 A_0}{1+\eta}k_0^{*\beta},$$

or

$$k_0^* = \left(\frac{(1-\alpha)(1-\beta)A_0}{1+\eta}(1+\zeta)^{-1/(1-\beta)}\right)^{1/(1-\beta)}, \tag{8}$$

then for each t we have

$$k_t = \left((1+\zeta)^{\frac{1}{1-\beta}}\right)^t k_0^*,$$

so

$$\frac{k_{t+1}}{k_t} = (1+\zeta)^{\frac{1}{1-\beta}}.$$

That is, each period grows at the limiting rate.

Figure 3 shows how quickly the growth ratios k_{t+1}/k_t converge for different starting values k_0. The parameters are $\alpha = .8$, $\beta = .3$, $\zeta = 0.348$ (which corresponds to an annual rate of .75% for a 40-year period), $\eta = 0.489$ (which corresponds to an annual rate of 1% for a 40-year period), $A_0 = 1$, and initial capital/labor ratios of $k_0 = 0.01855$ for the red dots, $k_0 = 10$ for the green dots, and $k_0 = .001$ for the blue dots. For these values, the limiting ratio is $(1+\zeta)^{1/(1-\beta)} = 1.53261$. More interesting, (7) also implies that the capital/labor ratios themselves converge, as is shown in Figure 4.

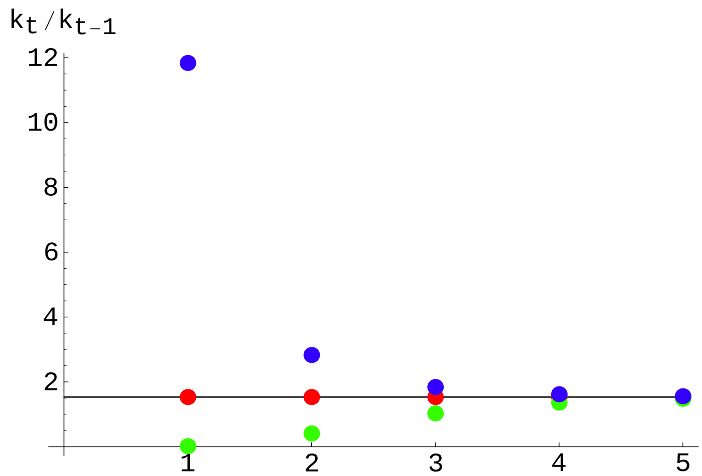

Figure 3. Convergence of k_{t+1}/k_t for different initial values k_0.

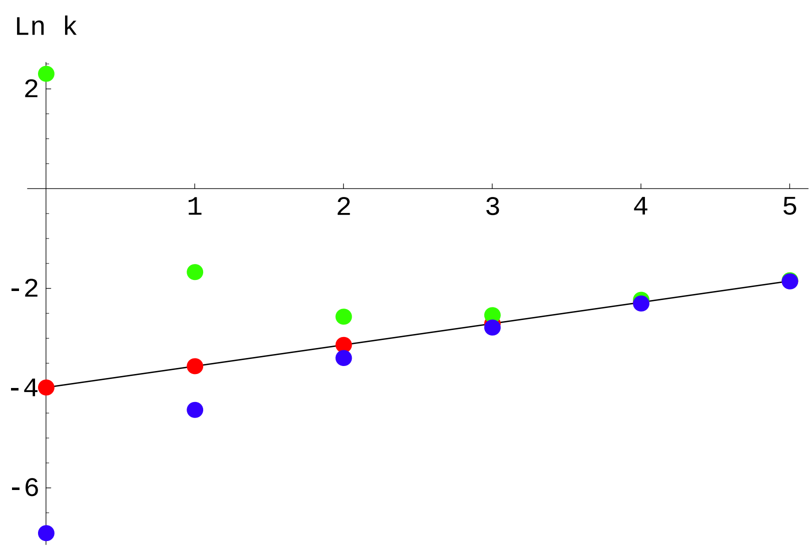

Figure 4. Convergence of capital/labor ratios for different initial values k_0.

However, do not think that saving rates and population growth rates do not matter. They do not affect the asymptotic growth *rate*, but they do affect the levels of k_t. Figure 5 shows the effect of decreasing the marginal propensity to consume (α) from .8 (red) to .7 (blue). A decrease in the population growth rate has an effect in the same direction.

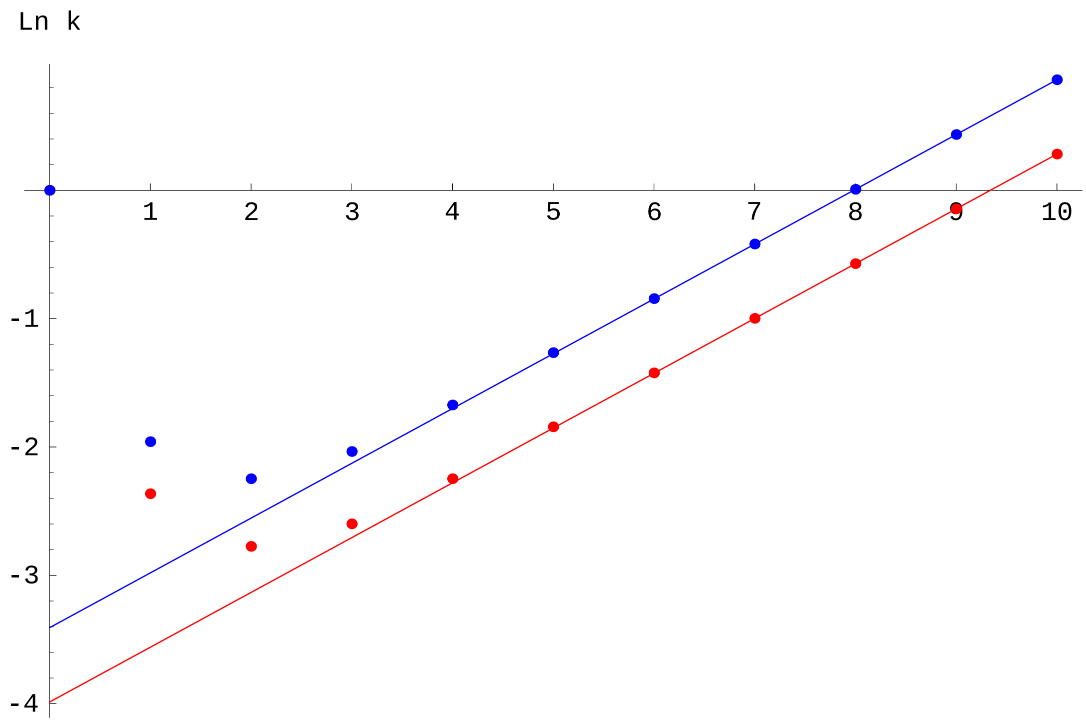

Figure 5. Effect of change in α on levels of k_t, but not growth rates.

As an aside, we can actually solve the difference equation for k_t, or more easily, for $\ln k_t$. From (6), we have

$$\ln k_{t+1} = \big[\ln(1-\alpha) + \ln A_0 + \ln(1-\beta) - \ln(1+\eta)\big] + t\ln(1+\zeta) + \beta \ln k_t,$$

which has the form

$$u_{t+1} = A + Bt + \beta u_t.$$

Iterating as above, we get

$$\begin{aligned}
u_{t+1} &= A + Bt + \beta u_t \\
&= A + Bt + \beta\left(A + B(t-1) + \beta u_{t-1}\right) \\
&= A(1+\beta) + B\big(t + \beta(t-1)\big) + \beta^2 u_{t-1} \\
&= A(1+\beta) + B\big(t + \beta(t-1)\big) + \beta^2\left(A + B(t-2) + \beta u_{t-2}\right) \\
&= A(1+\beta+\beta^2) + B\big(t + \beta(t-1) + \beta^2(t-2)\big) + \beta^3 u_{t-2} \\
&\ \ \vdots \\
&= A\sum_{n=0}^{t}\beta^n + B\left(\sum_{n=0}^{t}\beta^n(t-n)\right) + \beta^{t+1}u_0 \\
&= A\sum_{n=0}^{t}\beta^n + Bt\sum_{n=0}^{t}\beta^n - B\sum_{n=0}^{t}n\beta^n + \beta^{t+1}u_0 \\
&= (A+Bt)\left(\frac{1-\beta^{t+1}}{1-\beta}\right) - B\sum_{n=0}^{t}n\beta^n + \beta^{t+1}u_0.
\end{aligned}$$

Now, you probably don't know this off the top of your head, but maybe you have a Chemical Rubber Company *Handbook* or access to Mathematica, or are really good at math, so you could find out or figure out that

$$\begin{aligned}
\sum_{n=0}^{t} n\beta^n &= \frac{\beta(1-\beta^t - t\beta^t + t\beta^{t+1})}{(1-\beta)^2} \\
&= \frac{\beta}{(1-\beta)^2} + \beta^{t+1}\frac{1+t-t\beta}{(1-\beta)^2}
\end{aligned}$$

so

$$u_{t+1} = (A+Bt)\left(\frac{1}{1-\beta}\right) - B\frac{\beta}{(1-\beta)^2} + \beta^{t+1}\left(u_0 - \frac{A+Bt}{1-\beta} - \frac{1+t-t\beta}{(1-\beta)^2}\right)$$

or shifting indexes and regrouping,

$$u_t = \frac{B}{1-\beta}t + \frac{A}{1-\beta} - \frac{B}{(1-\beta)^2} + \beta^{t+1}\left(u_0 - \frac{A+Bt}{1-\beta} - \frac{1+t-t\beta}{(1-\beta)^2}\right).$$

Since $t\beta^t \to 0$ as $t \to \infty$, we see that u_t converges pointwise to the line

$$\frac{B}{1-\beta}t + \frac{A}{1-\beta} - \frac{B}{(1-\beta)^2}$$

as $t \to \infty$. So exponentiating,

$$\begin{aligned}
k_t &\to \left(\frac{(1-\alpha)(1-\beta)A_0}{1+\eta}(1+\zeta)^{-1/(1-\beta)}\right)^{1/(1-\beta)}\left((1+\zeta)^{1/(1-\beta)}\right)^t \\
&= k_0^*\left((1+\zeta)^{1/(1-\beta)}\right)^t,
\end{aligned}$$

where k_0^* is given by (8). This certainly explains why $k_{t+1}/k_t \to (1+\zeta)^{1/(1-\beta)}$.

References

[1] A. J. Auerbach and L. J. Kotlikoff. 1999. *Macroeconomics: An integrated approach*, 2d. ed. Cambridge, Mass.: MIT Press.